The Continuum Version of ϕ^4_{1+1}-theory in Light-Front Quantization

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Abstract

A genuine continuum treatment of the massive ϕ^4_{1+1}-theory in light-cone quantization is proposed. Fields are treated as operator valued distributions thereby leading to a mathematically well defined handling of ultraviolet and light cone induced infrared divergences and of their renormalization. Although non-perturbative the continuum light cone approach is no more complex than usual perturbation theory in lowest order. Relative to discretized light cone quantization, the critical coupling increases by 30% to a value $r = 1.5$. Conventional perturbation theory at the corresponding order yields $r_1 = 1$, whereas the RG improved fourth order result is $r_4 = 1.8 \pm 0.05$.

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1 Introduction

The discretized light front quantization (DLCQ) [1] has played an important role in clarifying infrared aspects of the theory which are decisive for the appearance of the vacuum sector field, the LC-counterpart of the nontrivial ground state of ET-quantization [2-7]. The popularity of DLCQ resides in the easy and conceptually simple treatment of the necessary infrared regularisation. However it has never been demonstrated that the limit where the periodicity length L goes to infinity is identical to the genuine continuum theory where momentum space discretization is avoided from the start. The reason lies in the infrared behaviour of the continuum theory which has not yet been understood. Our aim is to clarify this issue on the basis of a mathematically well defined procedure.

As an example we treat explicitely the ϕ^4_{1+1}-theory in the continuum and compare its results for the phase transition to the DLCQ case. It turns out that with the same type of physical approximations the characteristics of the phase transition are the same in both cases whereas the critical coupling strength and the dependence of the order parameter on the coupling strength are substantially different.

In connection with phase transitions there is a vital interest to dispose of a continuum version of the theory, if one is interested in the study of critical phenomena in the framework of effective theories. This is the point of view of statistical theories of fields in which cutoffs are introduced in order to define a momentum or mass scale below which the effective theory is valid. Critical points of phase transitions are determined from zeros of the β-function. To do this requires the complete knowledge of the cutoff dependence of the critical mass which can be given only by the continuum theory.

In Section 2 we make use of the concept of field operator valued distributions to have a mathematically well defined Fock expansion. This can be done in a chart-independent manner expressing the field as a surface integral over a manifold. Comparing the expressions for the Minkowski and light-front cases one sees that the regularization properties of the test functions are automatically transferred from the first to the second case ; thus it is ensured that, if the field is regular in the Minkowski case, it is also regular in the LC-case - it is the same field expressed by different surface integrals ! Actually what is called IR-divergence in the unregularized LC-field expansion is an UV-divergence in the LC-energy; it is only the special choice of coordinates which makes it look like an IR-divergence. Therefore there is no extra infrared (IR) singularity in the LC-case which would have to be treated separately : the ultraviolet (UV) behaviour of the field on the Minkowski manifold dictates the UV and IR behaviour on the LC manifold. There is absolutely no freedom in the LC-case beyond the choice of test functions relevant in the Minkowski case. Moreover, due to general properties known from functional analysis the independence of physical results from the special form of the test functions is ensured. In Section 3 we use the Haag expansion of field operators to define the decomposition into the particle sector field φ and the vacuum sector field Ω. In Section 4 we discuss the equation of motion (EM) for φ and the constraint for Ω which are coupled equations. Finally in Section 5 we discuss the so called mean field solution of these equations and the results for the phase transition. In Section 6 we rewrite the condition for the phase transition in the language of an effective theory and compare the results with the literature. In Appendix A we collect a number of results from the theory of distributions to substantiate the discussions in Section 2.

2 Fundamentals. Definitions and Conventions

The physical system under consideration is defined by the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2}\phi^2(x) - \frac{\lambda}{4!}\phi^4(x). \quad m^2 > 0 \quad ; \quad \lambda > 0. \tag{2.1}$$

From (2.1) follows the EM

$$\partial_\mu \partial^\mu \phi + m^2 \phi + \frac{\lambda}{3!}\phi^3 = 0. \tag{2.2}$$

The free field EM defines the free field $\varphi_0(x)$ which can be expanded in terms of free field creation and annihilation operators.

We start with the Minkowski case (ET quantization). The field $\varphi_0(x)$ is defined in the sense of distributions by the functional

$$\varphi_0\{u\} = (\varphi_0(x), u(x)) = \int d^4x \varphi_0(x) u(x) \tag{2.3}$$

where $u(x)$ is an element of the test function space in coordinate spaces. Plugging in the Fock expansion of $\varphi_0(x)$ yields

$$(\varphi_0(x), u(x)) = \frac{1}{(2\pi)^3} \int \frac{d^4x d^3p}{2\omega(\vec{p})}.[a(\vec{p})e^{-i<x,p>_M} + a^+(\vec{p})e^{+i<x,p>_M}].f(\omega(\vec{p}), \vec{p}) \tag{2.4}$$

where $< x, p >_M$ is the scalar product of x and p in the Minkowski metric, $\omega(\vec{p}) = \sqrt{m^2 + \vec{p}^2}$ is the on-shell energy and $f(p_0, \vec{p})$ is a test function in momentum space which is required to fall off sufficiently fast as a function of the arguments p_0, p_1, p_2, p_3 as any one of the $p_i' s$ goes to ∞.

The minimal conditions for the attenuation factor $f(p)$ are

$$\int \frac{d^3p}{2\omega(p)} |f(\omega(\vec{p}), \tilde{p})| | < s|a^+(\vec{p})|s' > | < \infty$$

and

$$\int \frac{d^3p}{2\omega(p)} |f(\omega(\tilde{p}), \vec{p})| | < r|a(\vec{p})|r' > | < \infty.$$

Here $(|s>, |s'>)$ and $(|r>, |r'>)$ are arbitrary pairs of states yielding nonvanishing matrix elements for a^+ and a respectively (see App. eq. (A.15)). This condition is necessary, if one wants to garantee that the Fourier integral in (2.4) is finite. $\varphi_0(x)$ is then defined as

$$\varphi_0(x) = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2\omega(\vec{p})} [a(\vec{p})e^{-i<x,p>_M} + a^+(\vec{p})e^{i<x,p>_M}] f(p_0, \vec{p}) \tag{2.5}$$

which can also be written as a surface integral over the manifold defined by $p_0^2 - \vec{p}^2 - m^2 = 0$ (see App. A). The positive and negative frequency parts in (2.4) and (2.5) are a consequence of the necessity to introduce two charts - corresponding to $p_0 = \pm\sqrt{p^2 + m^2}$ - if one wants to cover the manifold.

In the light-cone case the corresponding expression becomes

$$\varphi_0(x) = \frac{1}{(2\pi)^3}\int d^3\tilde{p}\frac{\theta(p^+)}{2p^+}[\tilde{a}(\vec{\tilde{p}})e^{-i<\tilde{x},\tilde{p}>_{LC}} + \tilde{a}^+(\vec{\tilde{p}})e^{i<\tilde{x},\tilde{p}>_{LC}}]\tilde{f}(p_0(\tilde{p}),\vec{p}(\tilde{p})). \tag{2.6}$$

Here $\tilde{x}$ and $\tilde{p}$ designate the light-cone variables

$$\begin{aligned}
\tilde{x}^0 &:= x^+ = \frac{1}{\sqrt{2}}(x^0 + x^3) \\
\tilde{x}^3 &:= x^- = \frac{1}{\sqrt{2}}(x^0 - x^3) \\
\tilde{x}^2 &:= x^i \quad , \quad i = 1,2.
\end{aligned} \tag{2.7}$$

$$\begin{aligned}
\tilde{p}^3 &:= p^+ = \frac{1}{\sqrt{2}}(p^0 + p^3) \\
\tilde{p}^0 &:= p^- = \frac{1}{\sqrt{2}}(p^0 - p^3) \\
\tilde{p}^i &= p^i \quad ; \quad i = 1,2
\end{aligned} \tag{2.8}$$

$< \tilde{x},\tilde{p} >_{LC}$ is the scalar product within the LC-metric. The creation and annihilation operators in (2.5) and (2.6) are related through

$$\tilde{a}(\vec{\tilde{p}}) = a(\vec{v}(\vec{\tilde{p}})) \;\; ; \;\; \tilde{a}^+(\vec{\tilde{p}}) = a^+(\vec{v}(\vec{\tilde{p}})) \;\; ; \;\; \forall\vec{\tilde{p}} \;\; |p^+ > 0 \tag{2.9}$$

where $\vec{v}(\vec{\tilde{p}})$ is the three vector defined by

$$\vec{v}(\vec{\tilde{p}}) = (\vec{p}_\perp, \frac{1}{\sqrt{2}}(p^+ - \frac{(p_\perp^2 + m^2}{2p^+})) \tag{2.10}$$

Finally $\tilde{f}(p_0(\vec{p}),\vec{p}(\tilde{p}))$ is the transformed test function.

Explicitely we have

$$\tilde{f} = f[\frac{1}{\sqrt{2}}(p^+ + \frac{p_\perp^2 + m^2}{2p^+}),(p_\perp^1,p_\perp^2,\frac{1}{\sqrt{2}}(p^+ - \frac{p_\perp^2 + m^2}{2p^+})] \tag{2.11}$$

from where it is clear that there is no infrared singularity in (2.6), if there is none in (2.5), since the singular behaviour of $\frac{1}{p^+}$ is completely damped out by the behaviour of the test function for $p^+ \to 0$. This is also clear from the fact that the two integrals in Eqs. (2.5) and (2.6) are equal (see App. A). Therefore, if $\varphi_0(x)$ is a bounded operator in the ET case it is also guaranteed to be bounded in the light cone case. Whereas the integral in the Minkowski case is composed of contributions from two charts, the final expression (2.6) in the LC-case goes only over one chart (the one with $p^+ > 0$). Originally there were also two charts, corresponding to $p^+ < 0$ and $p^+ > 0$, but the integral over the former one turns out to be equal to the integral over $p^+ > 0$ hence merging in a single expression ; this is due to the different topologies in the Minkowski and the LC-case : in the LC-case the sign of p^- is the same as the sign of p^+, whereas in the Minkowski case the sign of the energy is not correlated with signs of momentum components ; instead there is a sign ambiguity.

3 Decomposition of fields into particle sector and vacuum sector fields

In the DLCQ-case the total field $\phi(x)$ can be naturally decomposed into the particle sector $\varphi(x)$ - to be constructed from polynomials in $\tilde{a}^+(k^+)$ and $\tilde{a}(k^+)$ with total momentum $K^+ > 0$ - and the vacuum sector Ω - to be constructed from polynomials with total momemtum $K^+ = 0$ [4]. In the continuum case this decomposition can be achieved with the help of the Haag expansion [8].

As in the DLCQ-case we decompose $\phi(x) = \varphi(x) + \Omega$ where for fixed LC time we have :

$$\varphi(x) = \varphi_0(x) + \int dy_1^- dy_2^- \, g_2(x^- - y_1^-, x^- - y_2^-) : \varphi_0(y_1)\varphi_0(y_2) :$$
$$+ \int dy_1^- dy_2^- dy_3^- \, g_3(x^- - y_1^-, x^- - y_2^-, x^- - y_3) : \varphi_0(y_1)\varphi_0(y_2)\varphi_0(y_3) : + ... \tag{3.1}$$

All fields are taken at a fixed time, e.g. $x^+ = 0$; the argument y_i means therefore $y_i = (y_i^-, x^+)$.

Due to the properties of $\varphi_0(x)$ the support of φ in Fourier space is determined by the support of the test functions in $\varphi_0(x)$. The coefficient functions $g_2, g_3, , ...$ - or rather their Fourier transforms - have to be determined from the equation of motion and the constraints.

In order to obtain the vacuum sector field Ω - which by definition is x^- independent - we perform an additional integration over x^- and add a constant c-number part ϕ_0 :

$$\Omega := \phi_0 + \frac{1}{V} \int dx^- dy_1^- dy_2^- \, g_2(x^- - y_1^-, x^- - y_2^-) : \varphi_0(y_1)\varphi_0(y_2) :$$
$$+ \frac{1}{V} \int dx^- dy_1^- dy_2^- \, g_3(x^- - y_1^-, x^- - y_2^-, x^- - y_3^-) : \varphi_0(y_1)\varphi_0(y_2)\varphi_0(y_3) : + ..., \tag{3.2}$$

V being the integration volume.

Apparently the operator valued part of Ω is nonlocal. Due to the fact that φ_0 is defined as an operator valued distribution the integrations in (3.1) and (3.2) are well-defined.

Substituting the expansion (2.6) into the definition (3.2) one obtains after a lengthy but completely standard calculation the Fourier expansion of Ω :

$$\Omega = \phi_0 + \int_0^\infty \frac{dk^+}{4\pi k^+} f^2(k^+, \hat{k}^-(k^+)) C(k^+) \tilde{a}^+(k^+) \tilde{a}(k^+) + ... \tag{3.3}$$

where the coefficient $C(k^+)$ is given by

$$C(k^+) = \frac{2}{V} \int\int g_2(x^- - y_1^-, x^- - y_2^-) \cos[\frac{k^+}{2}(y_2 - y_1)] dy_1 dy_2$$

The higher terms of the expansion are not reproduced here because they will not be considered in this paper. Apparently one has as in DLCQ :

$$< 0|\Omega|0 > \;=\; < 0|\phi|0 >= \phi_0$$

and

$$\Omega|q_1^+, q_2^+, ...q_N^+ >= \lambda(q_1^+ ... q_N^+)|q_1^+, q_2^+, ...q_N^+ > \tag{3.4}$$

the eigenvalues $\lambda(q_1^+...q_N^+)$ being given by

$$\lambda(q_1^+...q_N^+) = \phi_0 + \sum_{i=1}^{N} \frac{f^2(q_i^+, \hat{q}^-(q_i^+))}{4\pi q_i^+} C(q_i^+). \tag{3.5}$$

This shows that within the bilinear approximation Ω acts like a momemtum dependent mass term.

4 Determination of ϕ_0 and $C(k^+)$

The field $\phi(x) = \varphi(x) + \Omega$ satisfies the LC-form of the equation of motion

$$2\partial_+\partial_-(\varphi(x) + \Omega) + m^2(\varphi(x) + \Omega) + \frac{\lambda}{3!}(\varphi(x) + \Omega)^3 = 0. \tag{4.1}$$

We first define an operator P which projects an operator $I\!F(x)$ onto the vacuum sector according to

$$P * I\!F(x) := \frac{1}{V} \int_{-\infty}^{+\infty} I\!F(x)dx. \tag{4.2}$$

Acting with P on (4.1) yields the constraint (the derivative term vanishes)

$$\theta_3 := m^2\Omega + \frac{\lambda}{3!}\Omega^3 + \frac{\lambda}{3!}\frac{1}{V} \int_{-\infty}^{+\infty} [\varphi^3(x)\Omega + \Omega + \varphi^2(x) + \varphi(x)\Omega\varphi(x)]dx = 0. \tag{4.3}$$

Projection with the complementary operator $Q := \mathbf{1} - P$ yields the equation of motion for $\varphi(x)$:

$$2\partial_+\partial_-\varphi(x) + m^2\varphi(x) + \frac{\lambda}{3!}Q * (\varphi(x) + \Omega)^3 = 0, \tag{4.4}$$

(4.3) and (4.4) are coupled operator valued equations which are solved by taking matrix elements between Hilbert space states. Technically this is very similar to the DLCQ case [4] :

One replaces $\varphi \to \varphi_0$ in (4.3) and (3.2) and calculates the matrix elements $< 0|\theta_3|0 >$ and $< k^+|\theta_3|k^+ >$.

The results are

$$< 0|\theta_3|0 >= \mu^2\phi_0 + \frac{\lambda}{3!}\phi_0^3 + \frac{\lambda}{24\pi} \int_0^{\infty} dk^+ \frac{C(k^+)\hat{f}^4(k^+)}{k^+} = 0. \tag{4.5}$$

and

$$\begin{aligned} < k^+|\theta_3|k^+ >= &\frac{\lambda}{6}C^3(k^+)\hat{f}^6(k^+) + \frac{\lambda}{2}\phi_0 C^2(k^+)\hat{f}^4(k^+) \\ &+[\mu^2\hat{f}^2(k^+) + \frac{\lambda}{2}\phi_0^2\hat{f}^2(k^+) + \frac{\lambda}{4\pi k^+}\hat{f}^4(k^+)]C(k^+) \\ &+ \frac{\lambda\phi_0}{4\pi k^+}\hat{f}^2(k^+) = 0 \quad ; \quad \forall \quad k^+ \end{aligned} \tag{4.6}$$

Here we use the notation $f(k^+, \hat{k}^-(k^+)) := \hat{f}(k^+)$. μ^2 is defined by

$$\mu^2 = m^2 + \frac{\lambda}{8\pi}\int_0^\infty \frac{dk^+}{k^+}\hat{f}^2(k^+) \tag{4.7}$$

which is nothing but the tadpole renormalization of the mass.In order to keep things as close as possible to the DLCQ case we use dimensionless momenta which we measure in units of μ.

Apparently (4.5) is an equation for ϕ_0 as a functional of $C(k^+)$; in turn (4.6) determines $C(k^+)$ as a function of ϕ_0 in the form of a cubic equation. Whereas the "exact" solution of eqs. (4.5) and (4.6) has to found numerically, important qualitative features of the solution can be discussed analytically.

In the region where k^+ is very small but $\hat{f}(k^+) \approx 1$ the solution of eq. (4.6) is simply

$$C(k^+) = -\frac{\lambda\phi_0}{4\pi k^+}\frac{f^2(k^+)}{\frac{\lambda}{4\pi k^+}\cdot f^4(k^+)} \approx -\frac{\phi_0}{f^2(k^+)} \tag{4.8}$$

i.e. the order parameter ϕ_0 determines the infrared behaviour of the vacuum vector part of the field.

Using (4.8) in the small k^+ region in (4.5) yields an integrand which behaves in the IR as $\dfrac{-\lambda\phi_0}{24\pi}\hat{f}^2(k^+)/k^+$ which has, up to numerical factors, the same behavior as the tadpole contribution in eq. (4.7). Therefore, any divergence in (4.5) arising from sending cutoffs present in $\hat{f}(k^+)$ to infinity can be cured by an appropriate mass counterterm. It is important to note that this result is independent of the particular form chosen for the test function. In the UV region the integral in (4.5) causes no problem whatsoever, since for $k^+ \to \infty \quad C(k^+) \sim \frac{1}{k^+}$ which yields an integrand $\sim 1/{k^+}^2$.

Nevertheless, in order to be able to evaluate the integrals one has to make a choice for $\hat{f}(k^+) = f(k^+, \hat{k}^-(k^+))$; without the on-shell condition $\hat{k} = \hat{k}^-(k^+)$ the test functions depend on the two variables k^+ and k^-. f can be chosen to have compact support and to be unlimitedly differentiable. In our case the support is the interior of a circle of radius Λ - which later on will be identified with a cutoff - . A possible form is [9]

$$\rho(k^+, k^-) = \exp(\frac{1}{\Lambda^2})\exp[-\frac{1}{\Lambda^2 - ({k^+}^2 + {k^-}^2)}] \quad ; \quad {k^+}^2 + {k^-}^2 \leq \Lambda^2$$

$$= 0 \qquad {k^+}^2 + {k^-}^2 > \Lambda^2$$

From this function one can construct another one which has the property that it equals 1 inside the 2-sphere of radius $\Lambda - \epsilon$ and falls to zero within the interval $[\Lambda - \epsilon, \Lambda]$; ϵ can be chosen as small as one likes without affecting the C^∞-character of the function [9].
In order to make the comparison with the DLCQ-case we use from now on the convention $\hat{k}^- = m^2/k^+$ instead of $\hat{k}^- = \frac{m^2}{2k^+}$ (see section 2). This means the use of the factor $\frac{1}{2}$ in eqs (2.7) and (2.8) instead of $1/\sqrt{2}$.

Taking into account the on-shell condition we arrive at the situation depicted in fig. 1.

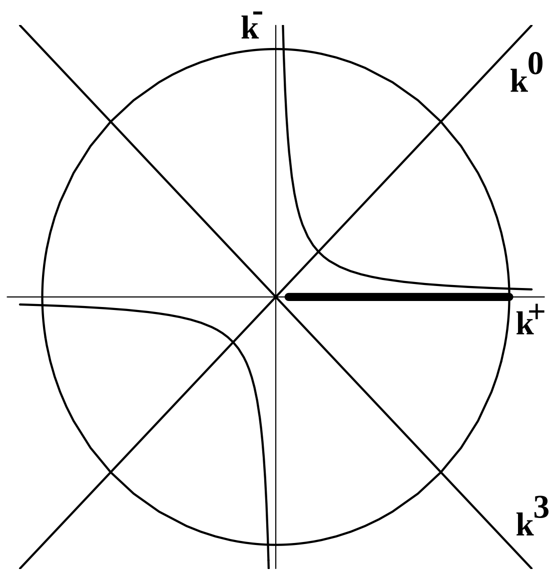

FIG. 1. The area inside the circle is the support of the test function $f(k^+, k^-)$. f equals unity inside a circle of radius $\Lambda - \epsilon$. The fall-off to zero takes place in the interval$[\Lambda - \epsilon, \Lambda]$. The hyperbola represents the on-shell condition $\hat{k}^- = m^2/k^+$. Its intersections with the circle determine the IR and UV cut-offs for the variable k^+. The resulting support for the function $\hat{f}(k^+)$ is indicated by the thick line. Only the right half is physically realized due to the kinematical condition $k^+ > 0$

The final result for $\hat{f}(k^+)$ is shown in fig. 2.

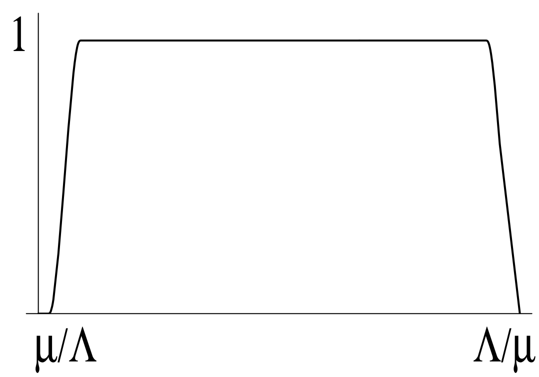

FIG. 2. Generic form of the test function$\hat{f}(k^+)$

In the limit where ϵ is arbitrarely small $\hat{f}$ acts like a cutoff at $1/\Lambda$ and Λ.

Using this cutoff form does not influence the physical results, since any other test function having the same support would yield the same results. Of course, instead of the divergent

integral in (4.9) we would get something else but the counter-term of eq. (4.8) would change correspondingly.We have tested and verified this statement by doing numerical integrations with small, but finite values of ϵ.

Using from now on the cutoff form for $\hat{f}$ and the dimensionless coupling $g = \lambda/4\pi\mu^2$ the integral to be renormalized becomes

$$I(\Lambda) = -\frac{g}{6}\int_{\mu/\Lambda}^{\Lambda/\mu} dk^+ \frac{C(k^+)}{k^+} \tag{4.9}$$

which is UV finite but IR divergent. Using the IR limit of $C(k^+)$ given in eq(4.8) $I(\Lambda)$ becomes

$$I(\Lambda) = -\frac{g\phi_0}{6}\log(\frac{\Lambda}{\mu}) - \frac{\phi_0}{6(\Lambda/\mu)} + 0(\frac{\mu^2}{\Lambda^2})... \quad . \tag{4.10}$$

Given that

$$< 0|\varphi_0^2|0 >= \frac{1}{8\pi}\int_{\mu/\Lambda}^{\Lambda/\mu} \frac{dk^+}{k^+} = \frac{1}{4\pi}\log(\Lambda/\mu) \tag{4.11}$$

the divergence in (4.9) can be compensated by the substraction of a mass-type counter-term $2\pi g\phi_0\varphi_0^2/3$.

5 Critical coupling and nature of the phase transition

In the vicinity of the phase transition where $\phi_0 \ll 1$ one can linearize eqs. (4.5) and (4.6) yielding

$$\begin{aligned} C(k^+) &= -\frac{\lambda\phi_0}{4\pi k^+}\frac{1}{\mu^2 + \frac{\lambda}{4\pi k^+}} \\ &= -g\phi_0\frac{1}{(g + k^+)}. \end{aligned} \tag{5.1}$$

The phase transition being determined by the vanishing of the mass term the critical coupling g_c is determined by the condition

$$1 = \frac{g_c^2}{6}\int_{\mu/\Lambda}^{\Lambda/\mu} \frac{dk^+}{k^+(k^+ + g_c)} - \frac{2\pi}{3}g_c < 0|\varphi_0^2|0 > \tag{5.2}$$

which follows from (4.5) after division by μ^2 and substraction of the mass counterterm. The integral in (5.2) can be evaluated analytically and yields with (4.11) :

$$\frac{g_c}{6}\log(g_c) = 1$$

which has the solution $g_c = 4.19...$

The corresponding value for DLCQ [4] is $g_{c_{DLCQ}} = 3.18$, i.e. there is a 30% deviation between the two cases. On the other hand there is no change in the nature of the phase transition (which is of second order) and of the critical exponents.

6 Comparison to theories of critical behaviour

In order to compare our value for the critical coupling g_c to results obtained earlier in ET-quantization we have to rewrite the constraint (4.5). The most complete study of the critical behaviour of ϕ^4_{1+1}-theory has been performed by Parisi [10] in the scenario of a theory of critical phenomena. In this context the field theory is interpreted as an effective theory with a cutoff Λ which defines the scale of validity of the theory. In the spirit of such a theory one has to keep in the constraints (4.5) the dependence on Λ and consider this equation as a prescription for the calculation of the critical mass $M(\tilde{g}, \Lambda)$. From this quantity one obtains the β-function

$$\beta(\tilde{g}) = M\frac{\partial M}{\partial \tilde{g}}|_{\lambda,\Lambda}. \tag{6.1}$$

Here the definition of the coupling $\tilde{g}$ differs from our g - all momenta and masses are measured in units of Λ, distances are replaced by the dimensionless quantity $x\Lambda$ - . g and $\tilde{g}$ are related by

$$g = \frac{\lambda}{4\pi\mu^2} = \frac{\lambda}{4\pi\Lambda^2}\frac{\Lambda^2}{\mu^2} := \tilde{g}\frac{\Lambda^2}{\mu^2} \tag{6.2}$$

We consider the constraint θ_3 as an equation for M^2 :

$$M^2 = \mu^2 + \frac{\lambda}{24\pi}\int_{\mu/\Lambda}^{\Lambda/\mu}\frac{C(k^+)}{k^+}dk^+ \tag{6.3}$$

Satisfying the constraint $\theta_3 = 0$ amounts to $M^2(\Lambda, \tilde{g}) = 0$ which in turn means $\beta(\tilde{g}, \Lambda) = 0$ i.e. the condition which defines the phase transition via the fixed point of the β-function.

We define the critical mass μ_c by

$$0 = \mu_c^2 + \frac{\lambda}{24\pi}\int_{\mu_c/\Lambda}^{\Lambda/\mu_c}\frac{C(k^+)}{k^+}dk^+ = \mu_c^2 + \frac{\lambda}{24\pi}[\log\frac{g_c + k^+}{k^+}]|_{\mu_c/\Lambda}^{\Lambda/\mu_c} \tag{6.4}$$

Using $g_c = \tilde{g}_c\frac{\Lambda^2}{\mu_c^2}$ and the new variable $y(\tilde{g}_c) = \log(\frac{\Lambda}{\mu_c})^2$, we obtain in the limit of large Λ

$$ye^y = \frac{6}{\tilde{g}_c} \tag{6.5}$$

The numerical solution of eq. (6.5) is shown in fig.3 as a function of $\tilde{g}_c$.

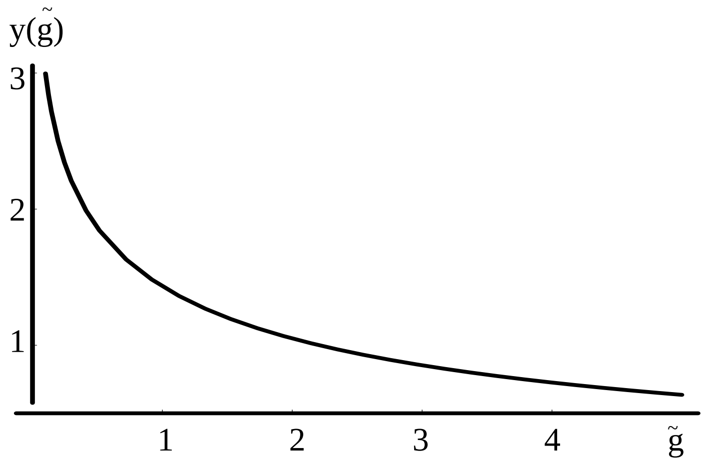

FIG. 3. The function $y(g)$ solution of eq. (6.5).

Knowing $y(\tilde{g}_c)$ we can now relate $\tilde{g}_c$ and g_c by

$$\tilde{g}_c = g_c(\frac{\mu_c}{\Lambda})^2 = g_c e^{-y(\tilde{g}_c)}$$

or

$$\tilde{g}_c e^{y(\tilde{g}_c)} = g_c \tag{6.6}$$

In fig. 4 the left hand side of eq. (6.6) is shown together with the straight line corresponding to $g_c = 4.19$.

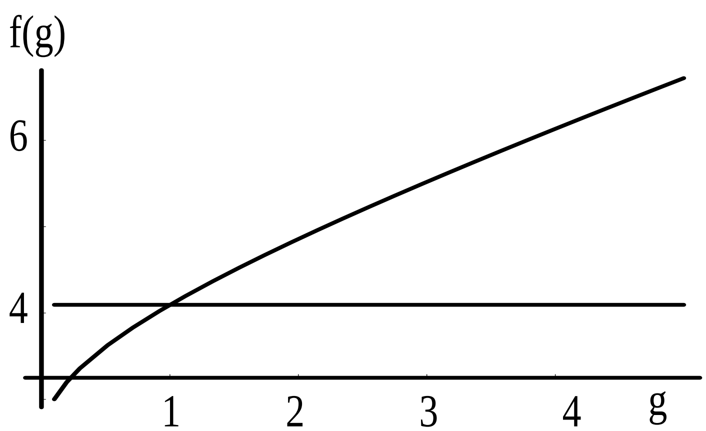

FIG. 4. The solution of eq. (6.6).

The numerical value for $\tilde{g}_c$ is 1.

Parisi [10] uses in his calculation still another coupling, called r, defined by $r = \frac{3\lambda}{8\pi\Lambda^2} = \frac{3}{2}\tilde{g}$. It is normalized in such way that the critical coupling at the order of one-loop is $r_1 = 1.0$.

He pushed his calculations up to four loops (with a Borel-improvement of the convergence of the asymptotic series) and obtained the value for r_4 reproduced in tab.1.

$\mathbf{r_1^{a,b}}$	$\mathbf{r_4^{a,b}}$	$\mathbf{r_{lat}^{c}}$	$\mathbf{r_{lc}}$
1	1.85	1.80 $\pm$ .05	1.5

a) Ref.[10]   b) Ref.[11]   c) Ref.[12]

TAB. 1. Critical couplings by different methods. r_{lc} is the value from the present continuous light-cone calculation

As far as the solution of the equation of motion is concerned our result corresponds to the 1-loop result of Parisi et al. (tadpole correction of the mass). On the other hand it is nonperturbative as far as the solution of the constraint is concerned. This is reflected by the considerable improvement relative to $r_1 = 1$ which brings us with $r_{lc} = 1.5$ already rather close to the four-loop result $r_4 = 1.85$ and to the lattice result $r_{lat} = 1.80 \pm 0.05$.

7 Conclusions

We have shown for scalar fields that the continuum quantum field theory quantized on the light-cone does not suffer from divergence problems beyond those present in conventional quantization, if the field operators are treated properly as operator valued distributions. The treatment is quite generic and should be rather easily generalizable to other types of fields. Apart from giving substantially different results for the critical coupling of the ϕ^4_{1+1}-theory as compared to the DLCQ, the continuum version has the advantage that it can be rewritten as an effective theory for critical phenomena. This is important for a detailed comparison with the literature because the most elaborate studies in conventional quantization have been performed in this domain. Our results compare very favorably with the best values of RG-improved fourth order perturbation theory and of lattice calculations which are reached up to 20%. Given the calculational simplicity of our approach - which on the technical level corresponds to first order perturbation theory - this is an encouraging success. It is attributed to the existence of an operator valued vacuum sector field which has to be added to the usual particle sector field and which is the LC-signature of nonperturbative physics.

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Appendix A

In this appendix we review some concepts from the theory of distributions which are at the basis of the results of Section 2.

A.1 - Pullback of distributions

Here we can give only a very short version. For more details the reader can consult e.g. reference [13].

We consider 2 open subsets U and V of R^n : $U, V \subset \mathcal{R}^n$ and a C^∞-diffeomorphism κ between them :

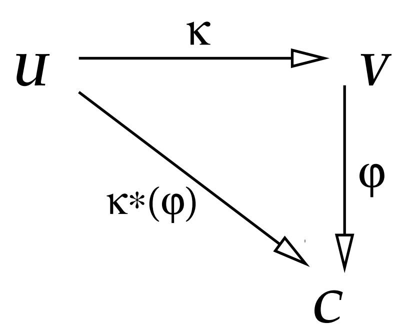
$$\kappa : U \rightarrow V.$$

Given a distribution

$$(\varphi, f) = \int_V \varphi(x) f_V(x) dx \tag{A.1}$$

with $f_V(x)$ a test function of compact support on V the distribution φ can be "pulled back" to U with the help of the pullback mapping $\kappa^* = \kappa \circ \varphi = \varphi(\kappa(x))$ (see fig.)

$$(\kappa^*[\varphi], f) = \int_U \varphi(\kappa(x)) f_U(x) dx \tag{A.2}$$



$$\kappa^* : C(V) \rightarrow C(U) \quad ; \quad \kappa^*(\varphi) := \varphi \circ \kappa$$

Introducing the inverse mapping

$$\tau := \kappa^{-1} : V \rightarrow U$$

and making the coordinate transformation $\xi := \kappa(x)$ (A.2) goes over into

$$(\kappa^*[\varphi], f) = \int_V \varphi(\xi) f(\tau(\xi)) |\det D(\tau(\xi))| d\xi \tag{A.3}$$

where $D(\tau(\xi))$ is the Jacobian of the coordinate transformation. (A.2) assigns to each distribution φ on V a "pulled back" distribution $\kappa^*[\varphi]$ on U and (A.3) is the prescription

for its evaluation in terms of φ. The applications which one has in mind in connection with (A.2) and (A.3) is the frequent case where one works with distributions which depend on functions of the integration variables, e.g. $\delta(p^2 - m^2)$.

In order to treat such a case in the pullback framework we take :
$U \subset \mathcal{R}^n, n > 1$; $V \subset \mathcal{R}^1$ is obtained from U through the mapping $Q : U \to V$ with a C^∞-function which we call now Q instead of κ. Moreover we introduce the (n-1)-dimensional sub-manifold Σ defined by $Q(x) = 0$;

$$\Sigma := \{x \in U | Q(x) = 0\}$$

and a Dirac δ^1-distribution δ_0 in $\mathcal{R}^1$.

The following theorem can be proved [13] :
If $\bigtriangledown Q(x) \neq 0 \quad \forall x \in U$ then the pullback of δ^1 exists and is defined by

$$\delta \circ Q = \frac{1}{| \bigtriangledown Q|} \delta_\Sigma, \tag{A.4}$$

where δ_Σ is a distribution defined by

$$(\delta_\Sigma, f) = \int_\Sigma f dS \quad \forall f \in U, \tag{A.5}$$

here dS is the euclidian surface measure on Σ ; this yields the distribution $\delta o Q$ as a surface integral :

$$(\delta \circ Q, f) = \int_\Sigma \frac{f}{| \bigtriangledown Q|} dS \tag{A.6}$$

Specializing Q to $Q(p) = p^2 - m^2$, p being the energy-momemtum 4-vector and using on Σ the charts

$$\Omega_\pm(\pm\omega(\vec{p}), \vec{p})$$

the distribution (A.6) can be rewritten as (in Minkowsky space) :

$$(\delta \circ Q, f) = \int \frac{d^3p}{2\omega(\vec{p})} [f(\omega(\vec{p}), \vec{p}) + f(-\omega(\vec{p}), \vec{p})] \tag{A.7}$$

where $\omega(\vec{p}) = +\sqrt{p^2 + m^2}$.

We introduce the two tempered distributions

$$\delta_\pm(p^2 - m^2) = 1(\vec{p})\theta(\pm p^0)\delta(p^2 - m^2), \tag{A.8}$$

where $1(\vec{p})$ is the unit distribution defined by

$$(1, f) = \int f(\vec{p}) d^3p,$$

and $\delta_\pm$ obey

$$(\delta_\pm, f) = \int \frac{d^3p}{2\omega(\vec{p})} f(\pm\omega(\vec{p}), \vec{p}). \tag{A.9}$$

Comparison with (A.7) shows that

$$(\delta \circ Q)(p) = \delta_+(p^2 - m^2) + \delta_-(p^2 - m^2).$$

Going back to (A.6) in the form

$$(\delta_+(p^2 - m^2) + \delta_-(p^2 - m^2), f(p)) = (\delta \circ Q, f(p)) = \int_{\Sigma^+} \frac{f}{|\nabla Q|} ds + \int_{\Sigma^-} \frac{f}{|\nabla Q|} ds$$

we see that δ_+ and δ_- can be written as surface integrals.

$$(\delta_\pm(p^2 - m^2), f(p)) = \int_{\Sigma^\pm} \frac{f}{|\nabla Q|} ds. \tag{A.10}$$

Here the two integrals $\int_{\Sigma^\pm} ds$ are over the two surfaces defined by the two signs of p_0 in $p_o = \pm\sqrt{\vec{p}^2 + m^2}$ with charts $\Omega^\pm$ (A.5). It is important to note that the integrals in (A.10) are independent of the special choice of the charts $\Omega^\pm$ which one makes on the surfaces $\Sigma^\pm$.

A.2 - Solutions of the KG-equation in Minkowski space

The tempered distribution $\chi(p)\delta(p^2 - m^2)$ defined by

$$\chi(p)\delta(p^2 - m^2) = \chi_+(p)\delta_+(p^2 - m^2) + \chi_-(p)\delta_-(p^2 - m^2) \tag{A.11}$$

satisfies

$$(\chi(p)\delta(p^2 - m^2), f(p)) = \int_\Sigma \frac{\chi f}{| \bigtriangledown Q|} ds \tag{A.12}$$

and solves the KG equation in momentum space :

$$(p^2 - m^2)v(p) = 0$$

with

$$v(p) = v_1(\vec{p})\theta(p^0)\delta(p^2 - m^2) + v_2(\vec{p})\theta(-p^0)\delta(p^2 - m^2).$$

From the distribution $\chi(p)\delta(p^2 - m^2)$ one obtains the solution of the coordinate space KG-equation as a distribution $\phi(x)$ defined by (Minkowsky metric)

$$\phi(x) := 2\pi\mathcal{F}_{M_x}[\chi(p)\delta(p^2 - m^2)] \tag{A.13}$$

where $\mathcal{F}_\mathcal{M}$ symbolises the Fourier-transform with Minkowsky metric, or more explicitely

$$(\phi(x), f(x)) = 2\pi(\chi(p)\delta(p^2 - m^2), \mathcal{F}_M(f)(p))$$

$$= 2\pi \int_\Sigma \frac{\chi(\mathcal{F}_M f)}{| \bigtriangledown Q|} ds$$

$$= 2\pi \int \frac{d^3p}{2\omega(\vec{p})} [\chi(\omega(\vec{p}), \vec{p}).(\mathcal{F}_M f)(\omega(\vec{p}), \vec{p}) + \chi(-\omega(\vec{p}), \vec{p})(\mathcal{F}_M f)(-\omega(\vec{p}), \vec{p})]. \tag{A.14}$$

In order to guarantee the existence of the two last integrals, one has to impose the condition on χ :

$$\int \frac{|\chi|}{|\bigtriangledown Q|} ds < \infty \quad \text{and} \quad \int \frac{d^3p}{2\omega(\vec{p})} |\chi(\pm\omega(\vec{p}), \vec{p})| < \infty. \qquad (A.15)$$

As it stands this is valid for classical fields. After quantization - where the $\chi's$ are replaced by creation and annihilation operators - (A.15) becomes a condition for matrix elements of these operators.

With (A.15) the functions

$$H_{\pm}(x,p) = \frac{1}{2\omega(\vec{p})} \chi(\pm\omega(\vec{p}), \vec{p}) e^{-i<x,\hat{p}_{\pm}>_M} f(x)$$

$$\hat{p}_{\pm} = (\pm\omega(\vec{p}), \vec{p})$$

are integrable ; consequently one can inject the Fourier representation of $\mathcal{F}_M f$ into (A.14) to obtain the regular distribution

$$(\phi(x), f(x)) = \frac{1}{(2\pi)^3} \int d^4x f(x) \int \frac{dp^3}{2\omega(\vec{p})} \; [\chi(\hat{p}_+) e^{-i<x,\hat{p}_+>_M} + \chi(\hat{p}_-) e^{-i<x,\hat{p}_->_M}] \quad (A.16)$$

From (A.16) one identifies the field $\phi(x)$ as

$$\phi = \frac{1}{(2\pi)^3} \int \frac{dp^3}{2\omega(\vec{p})} [\chi(\hat{p}_+) e^{-i<x,\hat{p}_+>_M} + \chi(\hat{p}_-) e^{-i<x,\hat{p}_->_M}] \qquad (A.17)$$

the chart independent form of (A.17) is

$$\phi = \frac{1}{(2\pi)^3} \int_{\Sigma} \frac{\chi(p)}{|\bigtriangledown Q(p)|} e^{-i<x,\hat{p}>_M} ds. \qquad (A.18)$$

A.3 - Solutions of the KG-equation on the LC

We go from the coordinates x, p to the corresponding LC coordinates $\tilde{x}, \tilde{p}$ via the mapping

$$\tilde{x} = \kappa \circ x \quad ; \quad \tilde{p} = \kappa \circ p$$

which leads to the LC-KG equation

$$(\Box_{LC} + m^2)\tilde{\phi}(\tilde{x}) = 0$$

with

$$\Box_{LC} = 2\partial_+\partial_- - \partial_\perp^2 \quad ;$$

Introducing the quadratic form

$$\tilde{Q}(\tilde{p}) := \tilde{p}^2 - m^2 \qquad (A.19)$$

one can define the distribution $\delta(\tilde{p}^2 - m^2)$ as the pullback of the δ-distribution under the mapping (A.19) :

$$\delta(\tilde{p}^2 - m^2) := \delta \circ \tilde{Q}$$

$$(\delta(\tilde{p}^2 - m^2), f(\tilde{p})) = \int_{\tilde{\Sigma}} \frac{f(\tilde{p})}{|\bigtriangledown \tilde{Q}|} d\tilde{s} \qquad (A.20)$$

where $\bigtriangledown\tilde{Q}(\tilde{p}) = 2(p^-, -\vec{p}_\perp, p^+) \neq 0$ on the mass shell. The manifold $\tilde{\Sigma}$ is a sum of two disconnected parts $\tilde{\Sigma}^+, p^+ > 0$ and $\tilde{\Sigma}^-, p^- < 0$.

Choosing on $\tilde{\Sigma}^\pm$ the charts

$$(\tilde{\Omega}^+ : (\tilde{\omega}(\vec{\tilde{p}}), \vec{\tilde{p}}) \quad ; \quad p^+ > 0)$$

and

$$(\tilde{\Omega}^- : (\tilde{\omega}(\vec{\tilde{p}}), \vec{\tilde{p}}) \quad ; \quad p^+ < 0)$$

$$\tilde{\omega}(\vec{\tilde{p}}) = \frac{p_\perp^2 + m^2}{2p^+}$$

one obtains :

$$\begin{aligned}(\delta(\tilde{p}^2 - m^2), f(\tilde{p})) &= \int_{\Sigma^+} \frac{f(\tilde{p})}{|\bigtriangledown \tilde{Q}|} d\tilde{s} + \int_{\Sigma^-} \frac{f(\tilde{p})}{|\bigtriangledown \tilde{Q}|} d\tilde{s} \\ &= \int d^3\tilde{p} \frac{\theta(p^+)}{2|p^+|} f(\tilde{\Omega}^+(\vec{\tilde{p}})) + \int d^3\tilde{p} \frac{\theta(-p^+)}{2|p^+|} f(\tilde{\Omega}^-(\vec{\tilde{p}})) \end{aligned} \tag{A.21}$$

The two integrals in (A.13) (with $\chi = 1$) and (A.20) can be shown to be equal. More generally it can be shown that for $\chi \in \mathcal{L}^1(\Sigma)$ and $\tilde{\chi} = \chi \circ u^{-1} \in \mathcal{L}^1(\tilde{\Sigma})$ one has the identity

$$\int_{\Sigma^\pm} \frac{\chi(p)}{|\bigtriangledown Q|} ds = \int_{\tilde{\Sigma}^\pm} \frac{\tilde{\chi}(\tilde{p})}{|\bigtriangledown \tilde{Q}|} d\tilde{s}. \tag{A.22}$$

Using this identity one arrives immediately at the result (see eq. (A.17)) :

$$\phi(x) = \frac{1}{(2\pi)^3} \int_\Sigma \frac{\chi(p)}{|\bigtriangledown Q(p)|} e^{-i<x,\hat{p}>_M} ds = \frac{1}{(2\pi)^3} \int_{\tilde{\Sigma}} \frac{\tilde{\chi}(\tilde{p})}{|\bigtriangledown \tilde{Q}(\tilde{p})|} e^{-i<\hat{x},\hat{p}>_L} d\tilde{s}. \tag{A.23}$$

Introducing the same charts as in (A.21) one obtains

$$\tilde{\phi}(\tilde{x}) = \frac{1}{(2\pi)^3} \int d^3p \frac{1}{2|p^+|} \tilde{\chi}(\tilde{\Omega}(\tilde{p})) e^{-i<\tilde{\Omega}(\tilde{p}),\tilde{x}>_L} \tag{A.24}$$

where

$$\tilde{\chi}(\tilde{\Omega}(\vec{\tilde{p}})) = \chi[\frac{1}{\sqrt{2}}(p^+ + \frac{p_\perp^2 + m^2}{2p^+}), \vec{p}_\perp, \frac{1}{\sqrt{2}}(p^+ - \frac{p_\perp^2 + m^2}{2p^+})].$$

The transition to the quantized field is made in (A.17) by the substitution

$$\chi(\hat{p}_+) \to a(\vec{p})$$
$$\chi(-\hat{p}_-) \to a^+(\vec{p})$$

yielding

$$\hat{\phi}(x) = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2\omega(\vec{p})} [a(p) e^{-i<x,\hat{p}_->_M} + a^+(\vec{p}) e^{i<x,\hat{p}_+>_M}].$$

In the LC-case the two contributions from $\tilde{p}^+ > 0$ and $\tilde{p}^+ < 0$ in (A.23) can be recast into a single one by first changing the integration variable in the second term $\vec{\tilde{p}} \to -\vec{\tilde{p}}$ and then making the substitution

$$\tilde{\chi}(\tilde{\Omega}(\vec{\tilde{p}})) \to \tilde{a}(\vec{\tilde{p}}) \;\; ; \;\; \tilde{p}^+ > 0$$

$$\tilde{\chi}(\tilde{\Omega}(-\vec{\tilde{p}})) \to \tilde{a}^+(\vec{\tilde{p}}) \;\; ; \;\; \tilde{p}^+ > 0$$

yielding

$$\hat{\tilde{\phi}}(\tilde{\chi}) = \frac{1}{(2\pi)^3} \int d^3\tilde{p} \frac{\theta(\tilde{p}^+)}{2\tilde{p}^+} [\tilde{a}(\vec{\tilde{p}}) e^{-i<\tilde{\Omega}(\tilde{p}),\tilde{x}>_L} + \tilde{a}^+(\vec{\tilde{p}}) e^{+i<\tilde{\Omega}(\tilde{p}),\tilde{x}>_L}]. \qquad (A.25)$$

For completeness we add a remark : In the literature [14] operator valued distributions have been introduced on the level of the Fock-space operators $a(p), a^+(p)$ by defining distributions

$$(a, f) = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2\omega(\vec{p})} a(\vec{p}) f(\omega(\vec{p}), \vec{p})$$

$$(a^+, f) = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2\omega(\vec{p})} a^+(\vec{p}) f(\omega(\vec{p}), \vec{p}) \qquad (A.26)$$

For the LC-case this reads

$$(\tilde{a}, \tilde{f}) = \frac{1}{(2\pi)^3} \int \frac{d^3\tilde{p}}{\tilde{p}^+} \tilde{a}(\vec{\tilde{p}}) \tilde{f}(\omega(\vec{p}), \vec{p})$$

$$(\tilde{a}^+, \tilde{f}) = \frac{1}{(2\pi)^3} \int \frac{d^3\tilde{p}}{\tilde{p}^+} \tilde{a}^+(\vec{\tilde{p}}) \tilde{f}(\omega(\vec{p}), \vec{p} \qquad (A.27)$$

Decomposing the field $\varphi_0(x)$ into positive and negative frequency parts $\varphi_0(x) = \varphi_0^+(x) + \varphi_0^-(x)$ we see that

$$\varphi_0^+(0) = (a, f) = (\tilde{a}, \tilde{f})$$
$$\varphi_0^-(0) = (a^+, f) = (\tilde{a}^+, \tilde{f})$$

This shows again that the field $\varphi_0(x)$ is well defined in the sense of operator valued distributions.

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